Flow on the symmetry plane
d of a total cavo-pulmonary connection

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1. Introduction

The flow inside a total cavo-pulmonary connection, a bypass operation of the right heart adopted in
presence of congenital malformation, is here studied for a specific geometry which has been recently
introduced in clinics (Amodeo et al. 1997). The analysis has been performed by preliminary experimental
observation and a novel symmetry-plane Navier-Stokes formulation. This method, once some basic
hypotheses are verified, allows to reproduce the flow on the symmetry plane of a three-dimensional field
by using an extension of the two-dimensional approach. The objective is to verify the presence of a central
vortex observed experimentally (Grigioni, Amodeo et al. 2000), and to analyse its stability and dissipative
properties. The topological changes and energy dissipation has been analysed in both cases of unbalanced
and of balanced pulmonary artery and caval flows.

2. Methods

Assume a Cartesian system of coordinates \(\{x, y, z\}\) and \(z = 0\) as a symmetry plane of a domain bounded
from above and below at \(z = \pm h(x, y)\); for symmetry the components of velocity and vorticity fields are
odd or even functions of \(z\); the in-plane components of velocity are even function while the normal
component is odd and is given by \(gz\) where the function \(g(x, y) = \frac{\partial u}{\partial z} \bigg|_{0}\). The equation of motion near the
symmetry plane are obtained by expressing all variables in Taylor series and retaining the leading terms in
\(z\); in doing this we consider a vorticity-streamfunction formulation of the Navier-Stokes equations. The
only non-zero vorticity equation on the symmetry plane is:

\[
\frac{\partial \omega}{\partial t} + u_x \frac{\partial \omega}{\partial x} + u_y \frac{\partial \omega}{\partial y} - g \omega = \nu \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} \right);
\]

where it is noticeable the appearance of a stretching term \(g \omega\) otherwise not present in purely two-
dimensional flows, and \(\nu\) is the kinematic viscosity, while the term \(\frac{\partial^2 \omega}{\partial z^2}\) is often negligible. The 3D
continuity equation can be written as

\[
\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = -g;
\]

The symmetry plane velocity can be expressed in general as the sum of an irrotational contribution plus a
divergence-free one (Batchelor 1967)

\[
u_x(x, y) = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial y}, \quad u_y(x, y) = \frac{\partial \phi}{\partial y} + \frac{\partial \psi}{\partial x};
\]

where the streamfunction is related to vorticity in the standard way, and the potential from 3D continuity.
\[
\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega, \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -g; \tag{4}
\]

The flow is bounded from above and below at \( z = \pm h(x, y) \) and the conservation of mass over the fluid column gives the integrated continuity equation

\[
\frac{\partial}{\partial x}(hU_x) + \frac{\partial}{\partial y}(hU_y) = 0, \tag{5}
\]

where \( U_x \) and \( U_y \) are the velocity components averaged over the range of \( z \) between \( \pm h(x, y) \); if we introduce the basic hypothesis that the symmetry plane velocity is proportional to such average velocity

\[
u(x, y) = \kappa U(x, y), \tag{6}
\]

substitution of the hypothesis (6) in equation (5) gives the following elliptic equation for the potential

\[
\frac{\partial}{\partial x} \left( \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \phi}{\partial y} \right) = \frac{\partial h}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial h}{\partial y} \frac{\partial \psi}{\partial x} \tag{7}
\]

The system in then closed by equation (7). The in-plane pressure can be evaluated \textit{a posteriori} by taking the divergence of the three-dimensional Navier-Stokes equation to give on the symmetry plane.

### 2.1 Numerical method of solution

The numerical method is analogous to the \( \omega-\psi \) formulation to solve pure plane flow (Zovatto & Pedrizzetti 2001) and it contains just an additional step for the solution of the potential \( \phi \) with respect to the standard numerical method in the \( \omega-\psi \) formulation.

The space domain has been discretised with linear triangular finite elements and equations have been rewritten on the finite element mesh using a Galerkin residual procedure resulting in a second order accuracy in space; while the time has been discretised with a second order scheme fully implicit for the viscous term.

### 3. Conclusions

A novel method for the simulation of the Navier-Stokes flow on the symmetry plane of a three-dimensional field has been introduced. This technique has been shown to be appropriate when a self-similarity in the velocity profile normally to the symmetry plane holds; thus it cannot model rapidly diverging ducts or significant helical flows. However, in its range of applicability, it allows the use of rapid and accurate plane computations with three-dimensional conservation of mass and momentum.

This technique has been applied for the analysis of the fluid mechanics associated to a specific TCPC already adopted in the clinical practice (Amodeo et al. 1997). The analysis has been stimulated by the experimental observation of a central vortex at the connection between the two venae cavae and the pulmonary artery (Grigioni, Amodeo et al. 2000). The numerical results have confirmed this observation and clarified the role of this vortex.

There is not a real vortex at the connection; rather, a weak stagnating zone is found (see fig. 1), without vorticity at its centre, surrounded by the wall and by a separated vorticity layer. In such a structure the pressure does not show the typical drop present at the centre of a vortex, in this sense the vortex may be seen a weakly dissipative flow division structure. On another side the vortex is bounded by an associated shear layer that becomes unstable at moderately high Reynolds number creating a spontaneous unsteadiness of period of about four times smaller that normal heartbeat. The central circulation also creates a complex distribution of wall shear stress, with separating and stagnating zones, which may affect
the quality of flow structure interaction. Results have been confirmed and characterised at varying pulmonary and caval flow ripartition.

The limitation of a symmetry plane approximation does not allow to draw any definitive clinical conclusion. It gives a view of the phenomena at varying parameters and preliminary indications that must be verified by a three-dimensional approach for the cases of specific interest. An in vitro work is in progress focused on the unsteady phenomena detected here, a computational analysis is also possible although still challenging for the accuracy needed to resolve wall and shear layers which are the primary responsible of this unsteadiness.

Fig. 1. Numerical solution in correspondence of the experimental case, $Re = 700$, $r_{vc} = 60/40$, $r_{pa} = 50/50$. (a) Streamlines. (b) Vorticity field contours: levels from zero (dotted line) step $\pm 0.5$; positive levels (gray lines), negative levels (black lines).

REFERENCES


